OPTIMAL COVERS WITH HAMILTON CYCLES IN RANDOM GRAPHS

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ABSTRACT. A packing of a graph G with Hamilton cycles is a set of edge-disjoint Hamilton cycles in G. Such packings have been studied intensively and recent results imply that a largest packing of Hamilton cycles in $G_{n,p}$ a.a.s. has size $\lfloor \delta(G_{n,p})/2 \rfloor$. Glebov, Krivelevich and Szabó recently initiated research on the 'dual' problem, where one asks for a set of Hamilton cycles covering all edges of G. Our main result states that for $\frac{\log^{117} n}{n} \leq p \leq 1 - n^{-1/8}$, a.a.s. the edges of $G_{n,p}$ can be covered by $\lceil \Delta(G_{n,p})/2 \rceil$ Hamilton cycles. This is clearly optimal and improves an approximate result of Glebov, Krivelevich and Szabó, which holds for $p \geq n^{-1+\varepsilon}$. Our proof is based on a result of Knox, Kühn and Osthus on packing Hamilton cycles in pseudorandom graphs.

1. INTRODUCTION

Given graphs H and G, an H-decomposition of G is a set of edge-disjoint copies of H in G which cover all edges of G. The study of such decompositions forms an important area of Combinatorics but it is notoriously difficult. Often an H-decomposition does not exist (or it may be out of reach of current methods). In this case, the natural approach is to study the packing and covering versions of the problem. Here an H-*packing* is a set of edge-disjoint copies of H in G and an H-*covering* is a set of (not necessarily edge-disjoint) copies of H covering all the edges of G. An H-packing is *optimal* if it has the largest possible size and an H-covering is *optimal* if it has the smallest possible size. The two problems of finding (near) optimal packings and coverings may be viewed as 'dual' to each other.

By far the most famous problem of this kind is the Erdős-Hanani problem on packing and covering a complete r-uniform hypergraph with k-cliques, which was solved by Rödl [15]. In this case, it turns out that the (asymptotic) covering and packing versions of the problem are trivially equivalent and the solutions have approximately the same value.

Packings of Hamilton cycles in random graphs $G_{n,p}$ were first studied by Bollobás and Frieze. (Here $G_{n,p}$ denotes the binomial random graph on n vertices with edge probability p.) Recently, the problem of finding optimal packings of edge-disjoint Hamilton cycles in a random graph has received a large amount of attention, leading to its complete solution in a series of papers by several authors (see below for more details on the history of the problem). The size of a packing

of Hamilton cycles in a graph G is obviously at most $\lfloor \delta(G)/2 \rfloor$, and this trivial bound turns out to be tight in the case of $G_{n,p}$ for *any* p.

The covering version of the problem was first investigated by Glebov, Krivelevich and Szabó [7]. Note that the trivial bound on the size an optimal covering of a graph G with Hamilton cycles is $\lceil \Delta(G)/2 \rceil$. They showed that for $p \geq n^{-1+\varepsilon}$, this bound is approximately tight, i.e. in this range, a.a.s. the edges of $G_{n,p}$ can be covered with $(1+o(1))\Delta(G_{n,p})/2$ Hamilton cycles. Here we say that a property A holds a.a.s. (asymptotically almost surely), if the probability that A holds tends to 1 as n tends to infinity.

The authors of [7] also conjectured that the above trivial bound on the size of an optimal covering should hold for any $p \gg \log n/n$. Our main result confirms this conjecture, unless p tends to 0 or 1 rather quickly.

Theorem 1. *Suppose that $G \sim G_{n,p}$, where $\frac{\log^{117} n}{n} \leq p \leq 1 - n^{-1/8}$. Then a.a.s. the edges of G can be covered by $\lceil \Delta(G)/2 \rceil$ Hamilton cycles.*

Unlike the situation with the Erdős-Hanani problem, the packing and covering problems are not equivalent in the case of Hamilton cycles. However, they do turn out to be closely related, so we now summarize the known results leading to the solution of the packing problem for Hamilton cycles in random graphs. Here 'exact' refers to a bound of $\lfloor \delta(G_{n,p})/2 \rfloor$.

authors	range of p	
Bollobás [2]	$\delta(G_{n,p}) = 2$	exact
Bollobás & Frieze [4]	$\delta(G_{n,p})$ bounded	exact
Frieze & Krivelevich [5]	p constant	approx.
Frieze & Krivelevich [6]	$p = \frac{(1+o(1))\log n}{n}$	exact
Knox, Kühn & Osthus [10]	$p \gg \frac{\log n}{n}$	approx.
Ben-Shimon, Krivelevich & Sudakov [1]	$\frac{(1+o(1))\log n}{n} \leq p \leq \frac{1.02\log n}{n}$	exact
Knox, Kühn & Osthus [11]	$\frac{\log^{50} n}{n} \leq p \leq 1 - n^{-1/5}$	exact
Krivelevich & Samotij [12]	$\frac{\log n}{n} \leq p \leq n^{-1+\varepsilon}$	exact
Kühn & Osthus [14]	$p \geq 2/3$	exact

In particular, the results in [4, 11, 12, 14] (of which [11, 12] cover the main range) together show that for any p, a.a.s. the size of an optimal packing of Hamilton cycles in $G_{n,p}$ is $\lfloor \delta(G_{n,p}/2 \rfloor$. This confirms a conjecture of Frieze and Krivelevich [6] (a stronger conjecture was made in [5]).

The result in [14] is based on a recent result of Kühn and Osthus [13] which guarantees the existence of a Hamilton decomposition in every regular 'robustly expanding' digraph. The main application of the latter was the proof (for large tournaments) of a conjecture of Kelly that every regular tournament has a Hamilton decomposition. But as discussed in [13, 14], the result in [13] also has a number of further applications to packings of Hamilton cycles in dense graphs and (quasi-)random graphs.

Recall that the above results imply an optimal packing result for any p. However, for the covering version, we need p to be large enough to ensure the existence

of at least one Hamilton cycle before we can find any covering at all. This is the reason for the restriction $p \gg \log n/n$ in the conjecture of Glebov, Krivelevich and Szabó [7] mentioned above. However, they asked the intriguing question whether this might extend to p which is closer to the threshold $\log n/n$ for the appearance of a Hamilton cycle in a random graph.

The approximate cover result of Glebov, Krivelevich and Szabó [7] uses the approximate packing result in [10] as a tool. More precisely, their proof applies the result in [10] to obtain an almost optimal packing. Then the strategy is to add a comparatively small number of Hamilton cycles which cover the remaining edges. Instead, our proof of Theorem 1 is based on the main technical lemma (Lemma 47) of the exact packing result in [11]. This is stated as Lemma 17 in the current paper and (roughly) states the following: Suppose we are given a regular graph H which is close to being pseudorandom and a pseudorandom graph G_1, where G_1 is allowed to be surprisingly sparse compared to H. Then we can find a set of edge-disjoint Hamilton cycles in $G_1 \cup H$ covering all edges of H. Our proof involves several successive applications of this result, where we eventually cover all edges of $G_{n,p}$. In addition, our proof crucially relies on the fact that in the range of p we consider, there is a small but significant gap between the degree of the unique vertex x_0 of maximum degree and the other vertex degrees (and the same holds for the vertex of minimum degree). This means that for all vertices $x \neq x_0$, we can afford to cover a few edges incident to x more than once. The analogous observation for the minimum degree was exploited in [11] as well.

The result in [7] also holds for dense quasi-random graphs. It would be interesting to obtain such results for sparser quasi-random graphs too. In fact, the result in [11] does apply in a quasi-random setting (see Theorem 48 in [11]), but the assumptions are quite restrictive and it is not clear to which extent they can be used to prove results for (n, d, λ)-graphs, say.

This paper is organized as follows: In the next section, we collect several results and definitions regarding pseudorandom graphs, mainly from [11]. In Section 3, we apply Tutte's theorem to give results which enable us to add a small number of edges to certain almost-regular graphs in order to turn them into regular graphs (without increasing the maximum degree). Finally, in Section 4 we put together all these tools to prove Theorem 1.

2. Pseudorandom graphs

The purpose of this section is to collect all the properties of $G_{n,p}$ that we need for our proof of Theorem 1. Throughout the rest of the paper, we always assume that n is sufficiently large for our estimates to hold. In particular, some of our lemmas only hold for sufficiently large n, but we do not state this explicitly. We write $\log$ for the natural logarithm and $\log^a n$ for $(\log n)^a$. Given functions $f, g : \mathbb{N} \to \mathbb{R}$, we write $f = \omega(g)$ if $f/g \to \infty$ as $n \to \infty$. We denote the average degree of a graph G by $d(G)$.

We will need the following Chernoff bound (see e.g. Theorem 2.1 in [9]).

Lemma 2. *Suppose that $X \sim Bin(n, p)$. For any $0 < a < 1$ we have*

$$\mathbb{P}(X \leq (1 - a)\mathbb{E}X) \leq e^{-\frac{a^2}{3}\mathbb{E}X}.$$

The following notion was first introduced by Thomason [16].

Definition 3. Let $p, \beta \geq 0$ with $p \leq 1$. A graph G is (p, β)-*jumbled* if for all $S \subseteq V(G)$ we have

$$\left| e_G(S) - p\binom{|S|}{2} \right| < \beta|S|.$$

We will also use the following immediate consequence of Definition 3. Suppose that G is a (p, β)-jumbled graph and $X, Y \subseteq V(G)$ are disjoint. Then

(1) $$|e(X, Y) - p|X||Y|| \leq 2\beta(|X| + |Y|).$$

To see this, note that $e(X, Y) = e(X \cup Y) - e(X) - e(Y)$. Now (1) follows from Definition 3 by applying the triangle inequality.

The following notion was introduced in [11].

Definition 4. Let G be a graph on n vertices. For a set $T \subseteq V(G)$, let $\overline{d}_G(T) := \frac{1}{|T|} \sum_{t \in T} d_G(t)$ be the average degree of the vertices of T in G. Then G is *strongly 2-jumping* if for all $T \subseteq V(G)$ we have

$$\overline{d}_G(T) \geq \delta(G) + \min\{|T| - 1, \log^2 n\}.$$

Note that a strongly 2-jumping graph G is '2-jumping', i.e. it has a unique vertex of minimum degree and all other vertices have degree at least $\delta(G) + 2$.

The next definition collects (most of) the pseudorandomness properties that we need.

Definition 5. A graph G on n vertices is p-*pseudorandom* if all of the following hold:

(P1) G is $(p, 2\sqrt{np(1-p)})$-jumbled.
(P2) For any disjoint $S, T \subseteq V(G)$,
    (i) if $\left(\frac{1}{|S|} + \frac{1}{|T|}\right)\frac{\log n}{p} \geq \frac{7}{2}$, then $e_G(S, T) \leq 2(|S| + |T|)\log n$,
    (ii) if $\left(\frac{1}{|S|} + \frac{1}{|T|}\right)\frac{\log n}{p} \leq \frac{7}{2}$, then $e_G(S, T) \leq 7|S||T|p$.
(P3) For any $S \subseteq V(G)$,
    (i) if $\frac{\log n}{|S|p} \geq \frac{7}{4}$, then $e(S) \leq 2|S|\log n$,
    (ii) if $\frac{\log n}{|S|p} \leq \frac{7}{4}$, then $e(S) \leq \frac{7}{2}|S|^2 p$.
(P4) We have $np - 2\sqrt{np \log n} \leq \delta(G) \leq np - 200\sqrt{np(1-p)}$.
(P5) We have $\Delta(G) \leq np + 2\sqrt{np \log n}$.
(P6) G is strongly 2-jumping.

The following definition is essentially the same, except that some of the bounds are more restrictive.

Definition 6. A graph G on n vertices is *strongly p-pseudorandom* if all of the following hold:

(SP1) G is $(p, \frac{3}{2}\sqrt{np(1-p)})$-jumbled.

(SP2) For any disjoint $S, T \subseteq V(G)$,

(i) if $\left(\frac{1}{|S|} + \frac{1}{|T|}\right) \frac{\log n}{p} \geq \frac{7}{2}$, then $e_G(S,T) \leq \frac{3}{2}(|S| + |T|) \log n$,

(ii) if $\left(\frac{1}{|S|} + \frac{1}{|T|}\right) \frac{\log n}{p} \leq \frac{7}{2}$, then $e_G(S,T) \leq 6|S||T|p$.

(SP3) For any $S \subseteq V(G)$,

(i) if $\frac{\log n}{|S|p} \geq \frac{7}{4}$, then $e(S) \leq \frac{3}{2}|S| \log n$,

(ii) if $\frac{\log n}{|S|p} \leq \frac{7}{4}$, then $e(S) \leq 3|S|^2 p$.

(SP4) We have $np - 2\sqrt{np \log n} \leq \delta(G) \leq np - 200\sqrt{np(1-p)}$.

(SP5) We have $\Delta(G) \leq np + \frac{15}{8}\sqrt{np \log n}$.

(SP6) G is strongly 2-jumping.

The following lemma is an immediate consequence of Lemmas 9–11, 13 and 14 from [11].

Lemma 7. *Let $G \sim G_{n,p}$, where $48^2 \log^7 n/n \leq p \leq 1 - 36 \log^{\frac{7}{2}} n/\sqrt{n}$. Then G is strongly p-pseudorandom with probability at least $1 - 11/\log n$.*

The next observation shows that if we add a few edges at some vertex x_0 of a strongly pseudorandom graph such that none of these edges is incident to the unique vertex of minimum degree, then we obtain a graph which is still pseudorandom.

Lemma 8. *Suppose that G is a strongly p-pseudorandom graph with $p, 1 - p = \omega(1/n)$. Let y_1 be the (unique) vertex of minimum degree in G and let $x_0 \neq y_1$ be any other vertex. Let F be a collection of edges not contained in G which are incident to x_0 but not to y_1 and such that $|F| \leq \sqrt{np \log n}/8$. Then the graph $G + F$ is p-pseudorandom.*

Proof. Let $G' := G + F$. Clearly, (SP4) and (SP6) are not affected by adding the edges of F, so G' satisfies (P4) and (P6). The bound on $|F|$ together with (SP5) immediately imply that G' satisfies (P5).

We now show that G' satisfies (P1). Indeed, for any $S \subseteq V(G')$, (SP1) implies that

$$
\begin{aligned}
\left| e_{G'}(S) - p\binom{|S|}{2} \right| &\leq \left| e_{G'}(S) - e_G(S) \right| + \left| e_G(S) - p\binom{|S|}{2} \right| \\
&\leq |S| + \frac{3}{2}\sqrt{np(1-p)}|S| \leq 2\sqrt{np(1-p)}|S|.
\end{aligned}
$$

To check (P2), suppose that $S, T \subseteq V(G')$ are disjoint. Without loss of generality we may assume that $|S| \leq |T|$. First suppose $\left(\frac{1}{|S|} + \frac{1}{|T|}\right) \frac{\log n}{p} \geq \frac{7}{2}$. Then (i) of (SP2) implies that

$$
e_{G'}(S,T) \leq e_G(S,T) + |T| \leq \frac{3}{2}(|S| + |T|) \log n + |T| \leq 2(|S| + |T|) \log n,
$$

as required. Now suppose that $\left(\frac{1}{|S|} + \frac{1}{|T|}\right) \frac{\log n}{p} \leq \frac{7}{2}$. Then (ii) of (SP2) implies that

$$e_{G'}(S,T) \leq e_G(S,T) + |T| \leq |T|\,(6p|S| + 1) \leq 7|S||T|p.$$

So (ii) of (P2) holds. The proof that (P3) holds is essentially the same.   □

We say that a graph G on n vertices is *u-downjumping* if it has a unique vertex x_0 of maximum degree, and $d(x_0) \geq d(x) + u$ for all $x \neq x_0$. The following result follows from Lemma 17 in [11] by considering complements. The latter lemma in turn follows easily from Theorem 3.15 in [2].

Lemma 9. *Let* $G \sim G_{n,p}$ *with* $p, 1-p = \omega\left(\log n/n\right)$. *Then a.a.s.* G *is* $5\frac{\sqrt{np(1-p)}}{\log n}$-*downjumping.*

The next result is intuitively obvious, but due to possible correlations between vertex degrees, it does merit some justification.

Lemma 10. *Suppose that* $\log^2 n/n < p' \leq p \leq 1 - \log^2 n/n$, *that* $p' \leq 1/2$ *and that* $G \sim G_{n,p}$. *Let* H *be a random subgraph of* G *obtained by including each edge of* G *into* H *with probability* p'/p. *Then a.a.s.* G *contains a unique vertex* x_0 *of maximum degree and* x_0 *does not have minimum degree in* H.

Proof. Fix any $\varepsilon > 0$. Let A be the event that G contains a unique vertex x_0 of maximum degree and that $d_H(x_0) = \delta(H)$. Let $f := np' - \sqrt{np'\log\log n}$. Let B be the event that $\delta(H) \leq f$. Note that $H \sim G_{n,p'}$. So Corollary 3.13 of [3] implies that $\mathbb{P}(\overline{B}) \leq \varepsilon$. Let C be the event that G contains a unique vertex x_0 of maximum degree and that $d_H(x_0) \leq f$ and note that $A \cap B \subseteq C$. Note also that $\mathbb{P}(A) \leq \mathbb{P}(A \cap B) + \mathbb{P}(\overline{B}) \leq \mathbb{P}(C) + \varepsilon$. We say that a graph F on n vertices is *typical* if $\Delta(F) \geq np$ and there is a unique vertex of degree $\Delta(F)$. Now let D be the event that G is typical. Then Corollary 3.13 of [3] and Lemma 9 together imply that $\mathbb{P}(\overline{D}) \leq \varepsilon$. For any fixed graph F on n vertices, let E_F denote the event that $G = F$. Then $\mathbb{P}(C) \leq \varepsilon + \sum_{F:\ F\ \text{typical}} \mathbb{P}(C \mid E_F)\mathbb{P}(E_F)$. Suppose that E_F holds, where F is typical. Let $N := d_G(x_0)$ (note that E_F determines N and x_0). Whether the event C holds is now determined by a sequence of N Bernoulli trials, each with success probability p'/p. So let $X \sim \text{Bin}(N, p'/p)$. Then $\mathbb{E}(X) = N(p'/p) \geq p'n$, which implies that $f \leq \mathbb{E}(X)(1 - \sqrt{\log\log n/\mathbb{E}(X)})$. Then an application of Lemma 2 gives us

$$\mathbb{P}(C \mid E_F) = \mathbb{P}(X \leq f) \leq e^{-\log\log n/3} \leq \varepsilon.$$

So $\mathbb{P}(C) \leq 2\varepsilon$, which in turn implies that $\mathbb{P}(A) \leq 3\varepsilon$. Since ε was arbitrary, this implies the result.   □

Hefetz, Krivelevich and Szabó [8] proved a criterion for Hamiltonicity which requires only a rather weak quasirandomness notion. We will use a special case of their Theorem 1.2 in [8]. In that theorem, given a set S of vertices in a graph G, we let $N(S)$ denote the external neighbourhood of S, i.e. the set of all those vertices $x \notin S$ for which there is some vertex $y \in S$ with $xy \in E(G)$. Also, we

say that G is *Hamilton-connected* if for any pair x, y of distinct vertices there is a Hamilton path with endpoints x and y.

Theorem 11. *Suppose that G is a graph on n vertices which satisfies the following:*
(HP1) *For every $S \subseteq V(G)$ with $|S| \leq n/\sqrt{\log n}$, we have $|N(S)| \geq 20|S|$.*
(HP2) *G contains at least one edge between any two disjoint subsets $A, B \subseteq V(G)$ with $|A|, |B| \geq n/\log n$.*
Then G is Hamilton-connected.

Theorem 12. *Let $G \sim G_{n,p}$ with $\log^8 n/n \leq p \leq 1 - n^{-1/3}$, and let x_0 be a vertex of maximum degree in G. Then a.a.s. $G - x_0$ is Hamilton-connected.*

Proof. It suffices to check that $G - x_0$ satisfies (HP1) and (HP2). For p in the above range, these properties are well known to hold a.a.s. for G with room to spare and so also hold for $G - x_0$. For completeness we point out explicit references. To check (HP1), first note that Lemma 7 implies that G is p-pseudorandom. So Corollary 37 of [11] applied with $A_x := N_G(x) \setminus \{x_0\}$ now implies that (HP1) holds. (HP2) is a special case of Theorem 2.11 in [3] – the latter guarantees a.a.s. the existence of many edges between A and B. $\square$

3. Extending graphs into regular graphs

The aim of this section is to show that whenever H is a graph which satisfies certain conditions and G is a p-pseudorandom graph on the same vertex set which is edge-disjoint from H, then G contains a spanning subgraph H' whose degree sequence complements that of H, i.e. such that $H \cup H'$ is $\Delta(H)$-regular. The conditions on H that we need are the following:

- H has even maximum degree.
- H is $\sqrt{np}$-downjumping.
- H satisfies $\Delta(H) - \delta(H) \leq (np \log n)^{5/7}$.

In order to show this we will use Tutte's f-factor theorem, for which we need to introduce the following notation. Given a graph $G = (V, E)$ and a function $f : V \to \mathbb{N} \cup \{0\}$, an f-*factor* of G is a subgraph G' of G such that $d_{G'}(v) = f(v)$ for all $v \in V$. Our approach will then be to set $f(v) := \Delta(H) - d_H(v)$ and attempt to find an f-factor in the pseudorandom graph G. The following result of Tutte [17, 18] gives a necessary and sufficient condition for a graph to contain an f-factor.

Theorem 13. *A graph $G = (V, E)$ has an f-factor if and only if for every two disjoint subsets $X, Y \subseteq V$, there are at most*

$$\sum_{x \in X} f(x) + \sum_{y \in Y} (d(y) - f(y)) - e(X, Y)$$

connected components K of $G - X - Y$ such that

$$\sum_{x \in K} f(x) + e(K, Y)$$

is odd.

When applying this result, we will often bound the number of components K of $G - X - Y$ for which $\sum_{x \in K} f(x) + e(K, Y)$ is odd by the total number of components of $G - X - Y$. The next lemma (which is a special case of Lemma 20 in [11]) implies that there are at most $|X| + |Y|$ such components.

Lemma 14. *Let $G = (V, E)$ be a p-pseudorandom graph on n vertices with $pn \geq \log n$. Then for any nonempty $B \subseteq V$, the number of components of $G[V \setminus B]$ is at most $|B|$. In particular, G is connected.*

The following lemma guarantees an f-factor in a pseudorandom graph, as long as $\sum_{v \in V} f(v)$ is even, $f(v)$ is not too large and for all but at most one vertex $f(v)$ is not too small either. (Clearly, the requirement that $\sum_{v \in V} f(v)$ is even is necessary.)

Lemma 15. *Let $G = (V, E)$ be a p-pseudorandom graph on n vertices with $pn \geq \log^{21} n$, and let $f : V \to \mathbb{N} \cup \{0\}$ be a function such that $\sum_{v \in V} f(v)$ is even. Suppose that G contains a vertex x_0 such that $f(x_0)$ is even and such that*

$$f(x_0) \leq (np \log n)^{\frac{5}{7}} \quad and \quad \sqrt{np} \leq f(v) \leq (np \log n)^{\frac{5}{7}} \quad for \ all \quad v \in V \setminus \{x_0\}.$$

Then G has an f-factor.

Proof. Given two disjoint sets $X, Y \subseteq V$, we define $\alpha_f(X, Y)$ to be the number of connected components K of $G - X - Y$ such that

$$\sum_{x \in K} f(x) + e(K, Y)$$

is odd. We also define

$$\beta_f(X, Y) := \sum_{x \in X} f(x) + \sum_{y \in Y} (d(y) - f(y)) - e(X, Y).$$

By Theorem 13, it then suffices to prove that $\alpha_f(X, Y) \leq \beta_f(X, Y)$.

We will first show that $\alpha_f(X, Y) \leq |X| + |Y|$. If either X or Y is nonempty, this follows immediately from Lemma 14. If both X and Y are empty, then we must show that $\alpha_f(\emptyset, \emptyset) = 0$. But this holds since G is connected by Lemma 14, and $\sum_{x \in V} f(x)$ is even by hypothesis. Hence $\alpha_f(X, Y) \leq |X| + |Y|$ in all cases.

Thus it suffices to show that

$$(2) \qquad\qquad\qquad \beta_f(X, Y) \geq |X| + |Y|.$$

This holds if $X = Y = \emptyset$. So it remains to consider the following cases.

Case 1. $|X| = 1$.

Let x denote the unique vertex in X. Suppose first that $Y = \emptyset$. In this case Lemma 14 implies that $G - x = G - X - Y$ is connected. If $x = x_0$ then $\sum_{v \in V \setminus \{x\}} f(v) = \sum_{v \in V} f(v) - f(x)$ is even. Thus $\alpha_f(X, Y) = 0$ and so $\beta_f(X, Y) \geq \alpha_f(X, Y)$, as desired. If $x \neq x_0$ then $\beta_f(X, Y) = f(x) \geq \sqrt{np} \geq 1 \geq \alpha_f(X, Y)$, as desired.

Thus we may assume that $Y \neq \emptyset$. Then

$$
\begin{aligned}
\beta_f(X,Y) \quad &\geq \quad \sum_{y \in Y} (d(y) - f(y)) - |X||Y| \\
&\overset{(P4)}{\geq} \quad \left(np - 2\sqrt{np \log n} - (np \log n)^{\frac{5}{7}} \right) |Y| - |Y| \\
&\geq \quad \frac{np}{2} |Y| \geq |X| + |Y|
\end{aligned}
$$

and so (2) holds.

Case 2. $|X| > 1$ and $|Y| \leq \frac{1}{4}|X|(np)^{-\frac{3}{14}} \log^{-\frac{5}{7}} n$.

Since $\sum_{y \in Y} d(y) \geq e(X,Y)$ it follows that in this case we have

$$
\begin{aligned}
\beta_f(X,Y) &\geq \sum_{x \in X} f(x) - \sum_{y \in Y} f(y) \geq (|X| - 1)\sqrt{np} - |Y|(np \log n)^{\frac{5}{7}} \\
&\geq \frac{\sqrt{np}}{2}|X| - \frac{\sqrt{np}}{4}|X| \geq 2|X| \geq |X| + |Y|,
\end{aligned}
$$

and so (2) holds.

Case 3. $1 < |X| \leq \frac{n}{2}$ and $|Y| > \frac{1}{4}|X|(np)^{-\frac{3}{14}} \log^{-\frac{5}{7}} n$.

It follows by (P1) and (1) that

$$
e(X,Y) \leq p|X||Y| + 4\sqrt{np}(|X| + |Y|).
$$

Thus

$$
\begin{aligned}
\beta_f(X,Y) - \alpha_f(X,Y) \quad &\geq \quad \sum_{y \in Y} (d(y) - f(y)) - e(X,Y) - |X| - |Y| \\
&\overset{(P4)}{\geq} \quad \left(np - 2\sqrt{np \log n} - (np \log n)^{\frac{5}{7}} \right) |Y| - p|X||Y| - 5\sqrt{np}(|X| + |Y|) \\
(3) \qquad &\geq \quad \left(p(n - |X|) - 2(np \log n)^{\frac{5}{7}} \right) |Y| - 5\sqrt{np}|X| \\
&\geq \quad \left(\frac{np}{2} - 2(np \log n)^{\frac{5}{7}} \right) |Y| - 5\sqrt{np}|X| \\
&\geq \quad \frac{1}{4} \left(\frac{(np)^{\frac{11}{14}}}{2 \log^{\frac{5}{7}} n} - 22\sqrt{np} \right) |X| \geq 0,
\end{aligned}
$$

as desired.

Case 4. $|X| > \frac{n}{2}$ and $|Y| > \frac{1}{4}|X|(np)^{-\frac{3}{14}} \log^{-\frac{5}{7}} n$.

In this case we have

$$
n - |X| \geq |Y| \geq \frac{|X|}{4(np)^{\frac{3}{14}} \log^{\frac{5}{7}} n} \geq \frac{n^{\frac{11}{14}}}{8p^{\frac{3}{14}} \log^{\frac{5}{7}} n}.
$$

But as in the previous case, one can show that (3) still holds and so

$$
\begin{aligned}
\beta_f(X,Y) - \alpha_f(X,Y) &\geq \left(p(n - |X|) - 2(np\log n)^{\frac{5}{7}} \right) |Y| - 5\sqrt{np}|X| \\
&\geq \left(\frac{(np)^{\frac{11}{14}}}{8\log^{\frac{5}{7}} n} - 2(np\log n)^{\frac{5}{7}} \right) |Y| - 5\sqrt{np}|X| \\
&\geq \frac{(np)^{\frac{11}{14}}}{9\log^{\frac{5}{7}} n}|Y| - 5\sqrt{np}|X| \\
&\geq \left(\frac{(np)^{\frac{4}{7}}}{36\log^{\frac{10}{7}} n} - 5\sqrt{np} \right) |X| \geq 0,
\end{aligned}
$$

as desired.

This completes the proof of the lemma. $\qquad\square$

Corollary 16. *Let G be a p-pseudorandom graph on n vertices, where $pn \geq \log^{21} n$. Suppose that H is a graph on $V(G)$ which satisfies the following conditions:*

- *H is $\sqrt{np}$-downjumping.*
- *If x_0 is the unique vertex of maximum degree in H then $H - x_0$ and $G - x_0$ are edge-disjoint.*
- *$\Delta(H)$ is even.*
- *$\Delta(H) - \delta(H) \leq (np\log n)^{\frac{5}{7}}$.*

Then there exists a $\Delta(H)$-regular graph H' such that $H \subseteq H' \subseteq G \cup H$.

Proof. Define $f(v) := \Delta(H) - d_H(v)$ for all $v \in V(G)$. Then

$$
\sum_{v\in V} f(v) = n\Delta(H) - \sum_{v\in V} d_H(v),
$$

which is even. Moreover $f(x_0) = 0$ and our assumptions on H imply that

$$
\sqrt{np} \leq f(v) \leq \Delta(H) - \delta(H) \leq (np\log n)^{\frac{5}{7}}
$$

for all $v \in V \setminus \{x_0\}$. We may therefore apply Lemma 15 to find an f-factor G' in G. Then $H' := H \cup G'$ is a $\Delta(H)$-regular graph as desired. $\qquad\square$

4. Proof of Theorem 1

The main tool for our proof of Theorem 1 is the following result from [11, Lemma 47]. Roughly speaking, it asserts that given a regular graph H_0 which is contained in a pseudorandom graph G and given a pseudorandom subgraph G_0 of G which is allowed to be quite sparse compared to H_0, we can find a set of edge-disjoint Hamilton cycles in $H_0 \cup G_0$ which cover all edges of H_0. For technical reasons, instead of a single pseudorandom graph G_0, in its proof we actually need

to consider a union of several edge-disjoint pseudorandom graphs $G_1, \ldots, G_{2m+1}$, where m is close to $\log n$.

Lemma 17. *Suppose that $p_0 \geq \frac{\log^{14} n}{n}$ and $p_1 \geq \frac{(np_0)^{\frac{3}{4}} \log^{\frac{5}{2}} n}{n}$. Let $m := \frac{\log(n^2 p_1)}{\log \log n}$, and for all $i \in [2m+1]$ set $p_i := p_1$ if i is odd, and $p_i := 10^{10} p_1$ if i is even. Let G be a p_0-pseudorandom graph on n vertices. Suppose that $G_1, \ldots, G_{2m+1}$ are pairwise edge-disjoint spanning subgraphs of G such that each G_i is p_i-pseudorandom. Moreover, for all $i \in [2m + 1]$, let H_i be an even-regular spanning subgraph of G_i with $\delta(G_i) - 1 \leq d(H_i) \leq \delta(G_i)$. Suppose that H_0 is an even-regular spanning subgraph of G which is edge-disjoint from $\bigcup_{i=1}^{2m+1} H_i$. Then there exists a collection $\mathcal{HC}$ of edge-disjoint Hamilton cycles such that the union $HC := \bigcup \mathcal{HC}$ of all these Hamilton cycles satisfies $H_0 \subseteq HC \subseteq \bigcup_{i=0}^{2m+1} H_i$.*

The following lemma is a special case of Lemma 22(ii) of [11]. Given p_i-pseudorandom graphs G_i as in Lemma 17, it allows us to find the even-regular spanning subgraphs H_i required by Lemma 17.

Lemma 18. *Let G be a p-pseudorandom graph on n vertices such that $p, 1 - p = \omega\left(\log^2 n/n\right)$. Then G has an even-regular spanning subgraph H with $\delta(G) - 1 \leq d(H) \leq \delta(G)$.*

The next lemma ensures that $G \sim G_{n,p}$ contains a collection of Hamilton cycles which cover all edges of G except for some edges at the vertex x_0 of maximum degree and such that every edge at x_0 is covered at most once. Theorem 1 will then be an easy consequence of this lemma and Theorem 12.

Lemma 19. *Let $G \sim G_{n,p}$, where $\frac{\log^{117} n}{n} \leq p \leq 1 - n^{-\frac{1}{8}}$. Then a.a.s. G has a unique vertex x_0 of degree $\Delta(G)$ and there exist a collection $\mathcal{HC}$ of Hamilton cycles in G and a collection F of edges incident to x_0 such that*

(i) *every edge of $G - F$ is covered by some Hamilton cycle in $\mathcal{HC}$;*
(ii) *no edge in F is covered by a Hamilton cycle in $\mathcal{HC}$;*
(iii) *no edge incident to x_0 is covered by more than one Hamilton cycle in $\mathcal{HC}$.*

The strategy of our proof of Lemma 19 is as follows. We split $G \sim G_{n,p}$ into three edge-disjoint random graphs G_1, G_2 and R such that the density of G_1 is almost p and both G_2 and R are much sparser. It turns out we may assume that the vertex x_0 of maximum degree in G also has maximum degree in G_1. We then apply Corollary 16 in order to extend G_1 into a $\Delta(G_1)$-regular graph by using some edges of R. Next we apply Lemma 17 in order to cover this regular graph with edge-disjoint Hamilton cycles, using some edges of G_2.

Let H_2 be the subgraph of $R \cup G_2$ which is not covered by these Hamilton cycles. Again, we can make sure that x_0 is still the vertex of maximum degree in H_2. We now apply Corollary 16 again in order to extend H_2 into a $\Delta(H_2)$-regular graph H_2' by using edges of a random subgraph R' of G_1 (i.e. edges which we have already covered by Hamilton cycles). Finally, we would like to apply Lemma 17 in order to cover this regular graph by edge-disjoint Hamilton cycles, using edges of another sparse random subgraph G' of G_1. However, this means that in the last

step we might use edges of G' at x_0, i.e. edges which have already been covered with edge-disjoint Hamilton cycles. Clearly, this would violate condition (iii) of the lemma.

We overcome this problem as follows: at the beginning, we delete all those edges at x_0 from G_1 which lie in G', and then we regularize and cover the graph H_1 thus obtained from G_1 as before, instead of G_1 itself. However, we have to ensure that x_0 is still the vertex of maximum degree in H_1. This forces us to make G' quite sparse: the average degree of G' needs to be significantly smaller than the gap between $d_G(x_0) = \Delta(G)$ and the degree of the next vertex, i.e. significantly smaller than $\sqrt{np(1-p)}/\log n$. Unfortunately it turns out that such a choice would make G' too sparse to apply Lemma 17 in order to cover H_2. Thus the above two 'iterations' are not sufficient to prove the lemma (where each iteration consists of an application of Corollary 16 to regularize and then an application of Lemma 17 to cover). But with three iterations, the above approach can be made to work.

Proof of Lemma 19. Lemmas 7 and 9 imply that a.a.s. G satisfies the following two conditions:

(a) G is p-pseudorandom.

(b) G is $5u$-downjumping, where $u := \frac{\sqrt{np(1-p)}}{\log n}$.

Note that

$$(4) \qquad (np)^{\frac{27}{64}} \log^{\frac{259}{32}} n = \frac{\sqrt{np(1-p)}}{\log n} \cdot \frac{\log^{\frac{291}{32}} n}{(np)^{\frac{5}{64}}\sqrt{1-p}} \leq \frac{u}{2}.$$

Indeed, to see the last inequality note that either $1-p \geq 1/2$ and $(np)^{\frac{5}{64}} \geq \log^{\frac{292}{32}} n$ or $(np)^{\frac{5}{64}} \geq (n/2)^{\frac{5}{64}}$ and $\sqrt{1-p} \geq n^{-\frac{1}{16}}$. So here we use the bounds on p in the lemma. Define

$$p_2 \quad := \quad \frac{(np)^{\frac{3}{4}} \log^{\frac{7}{2}} n}{n} \geq \frac{\log^{91} n}{n},$$

$$p_3 \quad := \quad \frac{(np_2)^{\frac{3}{4}} \log^{\frac{7}{2}} n}{n} = \frac{(np)^{\frac{9}{16}} \log^{\frac{49}{8}} n}{n} \geq \frac{\log^{71} n}{n},$$

$$p_3' \quad := \quad 1600 p_3,$$

$$p_4 \quad := \quad \frac{(np_3)^{\frac{3}{4}} \log^{\frac{7}{2}} n}{n} = \frac{(np)^{\frac{27}{64}} \log^{\frac{259}{32}} n}{n} \geq \frac{\log^{57} n}{n},$$

$$p_1 \quad := \quad p - 2p_2 - p_3,$$

$$m_i \quad := \quad \frac{\log(n^2 p_i)}{\log\log n} \quad \text{for all} \quad 2 \leq i \leq 4,$$

$$p_{(i,j)} \quad := \quad \begin{cases} \frac{p_i}{(10^{10}+1)m_i+1} & \text{if } 2 \leq i \leq 4 \text{ and if } j \in [2m_i+1] \text{ is odd,} \\ \frac{10^{10}p_i}{(10^{10}+1)m_i+1} & \text{if } 2 \leq i \leq 4 \text{ and if } j \in [2m_i+1] \text{ is even.} \end{cases}$$

Now form random subgraphs of G as follows. First partition G into edge-disjoint random graphs G_1, G_2, G_3 and R_2 such that $G_i \sim G_{n,p_i}$ for $i = 1,2,3$ and

$R_2 \sim G_{n,p_2}$. (This can be done by randomly including each edge e of G into precisely one of G_1, G_2, G_3 and R_2, where the probability that e is included into G_i is p_i/p and the probability that e is included into R_2 is p_2/p, independently of all other edges of G.) We then choose edge-disjoint random subgraphs R'_2, R_4 and G_4 of G_1 with $R'_2 \sim G_{n,p_2}$, $R_4 \sim G_{n,p_4}$, and $G_4 \sim G_{n,p_4}$. (Since $p_1 \geq p_2 + 2p_4$ this can be done similarly as before.) Next we choose a random subgraph G'_3 of G_2 such that $G'_3 \sim G_{n,p'_3}$. To summarize, we thus have the following containments, where $\dot\cup$ denotes the edge-disjoint union of graphs:

$$G = G_1 \mathbin{\dot\cup} G_2 \mathbin{\dot\cup} G_3 \mathbin{\dot\cup} R_2 \quad \text{and} \quad G_1 \supseteq R'_2 \mathbin{\dot\cup} R_4 \mathbin{\dot\cup} G_4 \quad \text{and} \quad G_2 \supseteq G'_3.$$

Finally, for each $i \in \{2, 3, 4\}$, we partition G_i into edge-disjoint random subgraphs $G_{(i,1)}, \ldots, G_{(i,2m_i+1)}$ with $G_{(i,j)} \sim G_{n,p_{(i,j)}}$. Lemma 7 and a union bound implies that a.a.s. the following conditions hold:

(c) G_i is p_i-pseudorandom for all $i = 1, \ldots, 4$.
(d) $G_{(i,j)}$ is $p_{(i,j)}$-pseudorandom for all $i = 2, 3, 4$ and all $j \in [2m_i + 1]$.
(e) R_2 and R'_2 are p_2-pseudorandom, and R_4 is p_4-pseudorandom.
(f) $R_2 \cup G_2 \cup R'_2 \cup G_3$ is strongly $(3p_2 + p_3)$-pseudorandom and $G'_3 \cup G_3 \cup R_4 \cup G_4$ is strongly $(p'_3 + p_3 + 2p_4)$-pseudorandom.

Since $R_2 \cup G_2 \cup R'_2 \cup G_3 \sim G_{n,3p_2+p_3}$ and $G'_3 \cup G_3 \cup R_4 \cup G_4 \sim G_{n,p'_3+p_3+2p_4}$, Lemma 10 implies that a.a.s. the following condition holds:

(g) Let x_0 be the unique vertex of maximum degree of G. Then x_0 is not the vertex of minimum degree in $R_2 \cup G_2 \cup R'_2 \cup G_3$ or $G'_3 \cup G_3 \cup R_4 \cup G_4$.

It follows that a.a.s. conditions (a)–(g) are all satisfied; in the remainder of the proof we will thus assume that they are. We can apply Lemma 18 for each $i = 2, 3, 4$ and each $j \in [2m_i + 1]$ to obtain an even-regular spanning subgraph $H_{(i,j)}$ of $G_{(i,j)}$ with $\delta(G_{(i,j)}) - 1 \leq d(H_{(i,j)}) \leq \delta(G_{(i,j)})$.

As indicated earlier, our strategy consists of the following three iterations. The purpose of the first iteration is to cover all the edges of G_1. To do this, we will apply Corollary 16 in order to extend G_1 into a regular graph H'_1, using some edges of R_2. (Actually we will first set aside a set F_1 of edges of G_1 at x_0, but this will still leave x_0 the vertex of maximum degree in $H_1 := G_1 - F_1$. In particular, F_1 will contain the set F^* of all edges of G_4 at x_0.) We will then apply Lemma 17 to cover H'_1 with edge-disjoint Hamilton cycles, using some edges of G_2.

The purpose of the second iteration is to cover all the edges of $G_2 \cup R_2$ not already covered in the first iteration – we denote this remainder by H_2. It turns out that x_0 will still be the vertex of maximum degree in H_2. If $\Delta(H_2)$ is odd, then we will add one edge from $F_1 \setminus F^*$ to H_2 to obtain a graph H'_2 of even maximum degree. Otherwise, we simply let $H'_2 := H_2$. We extend H'_2 into a regular graph H''_2 using Corollary 16 and some edges of R'_2, then cover H''_2 with edge-disjoint Hamilton cycles using Lemma 17 and some edges of G_3.

The purpose of the third iteration is to cover all the edges of G_3 not already covered in the second iteration – we denote this remainder by H_3. We first add some (so far unused) edges from $F_1 \setminus F^*$ to H_3 in order to make x_0 the unique vertex of maximum degree. Let H'_3 denote the resulting graph. We then extend

H_3' into a regular graph H_3'' using Corollary 16 and some edges of R_4, and finally cover H_3'' with edge-disjoint Hamilton cycles using Lemma 17 and some edges of G_4.

Note that we did not use any edges of R_2' at x_0 when turning H_2' into H_2'' since x_0 is a vertex of maximum degree in H_2'. Similarly, we did not use any edges of R_4 at x_0 when turning H_3' into H_3''. Moreover, F^* was the set of all edges of G_4 at x_0 and no edge in F^* was covered in the first two iterations. Altogether this means that we do not cover any edge at x_0 more than once.

Note that in the second and third iterations, the graphs R_2' and R_4 we use for regularising consist of edges we have already covered. In the second iteration, this turns out to be a convenient way of controlling the difference between the maximum and minimum degree of H_3 (which might have been about $\Delta(G) - \delta(G)$ if we had used uncovered edges). In the third iteration, there are simply no more uncovered edges available.

After outlining our strategy, let us now return to the actual proof. We claim that x_0 is the unique vertex of maximum degree in G_1 and that G_1 is $4u$-downjumping. Indeed, for all $x \neq x_0$ we have

$$
\begin{aligned}
d_{G_1}(x) = d_G(x) - d_{G_2 \cup G_3 \cup R_2}(x) &\overset{(b)}{\leq} d_G(x_0) - 5u - d_{G_2 \cup G_3 \cup R_2}(x) \\
&= d_{G_1}(x_0) + d_{G_2 \cup G_3 \cup R_2}(x_0) - 5u - d_{G_2 \cup G_3 \cup R_2}(x) \\
&\leq d_{G_1}(x_0) + \Delta(G_2) + \Delta(G_3) + \Delta(R_2) - 5u - \delta(G_2) - \delta(G_3) - \delta(R_2) \\
&\leq d_{G_1}(x_0) - \left(5u - 12\sqrt{np_2 \log n} \right),
\end{aligned}
$$

where the last inequality follows from the facts that both G_2 and R_2 are p_2-pseudorandom, G_3 is p_3-pseudorandom, $p_3 \leq p_2$ as well as from (P4) and (P5). But

$$
(5) \qquad \sqrt{np_2 \log n} = (np)^{\frac{3}{8}} \log^{\frac{9}{4}} n \overset{(4)}{\leq} \frac{u}{2} \cdot (np)^{-\frac{3}{64}} \leq \frac{u}{\log n}.
$$

Altogether this shows that $d_{G_1}(x) \leq d_{G_1}(x_0) - 4u$ for all $x \neq x_0$. Thus G_1 is $4u$-downjumping and x_0 is the unique vertex of maximum degree in G_1, as desired. Note that

$$
(6) \qquad \Delta(G_4) \leq 2np_4 = 2(np)^{\frac{27}{64}} \log^{\frac{259}{32}} n \overset{(4)}{\leq} u.
$$

Let F^* be the set of all edges of G_4 which are incident to x_0. Thus $|F^*| \leq u$ by (6). Choose a set F_1 of edges incident to x_0 in G_1 such that $F^* \subseteq F_1$,

$$
(7) \qquad 3u - 1 \leq |F_1| \leq 3u
$$

and such that $\Delta(G_1 - F_1)$ is even. Note that we used (6) and thus the full strength of (4) (in the sense that it would no longer hold if we replace 117 by 116 in the lower bound on p stated in Lemma 19) in order to be able to guarantee that $F^* \subseteq F_1$. So this is the point where we need the bounds on p in the lemma. Let $H_1 := G_1 - F_1$. Thus H_1 is still u-downjumping.

Our next aim is to apply Corollary 16 in order to extend H_1 into a $\Delta(H_1)$-regular graph H_1', using some of the edges of R_2. So we need to check that the conditions in Corollary 16 are satisfied. But since G_1 is p_1-pseudorandom we have

$$\Delta(H_1) - \delta(H_1) \leq \Delta(G_1) - \delta(G_1) \overset{\text{(P4),(P5)}}{\leq} 4\sqrt{np_1 \log n}$$

$$(8) \qquad\qquad \leq 4\sqrt{np \log n} = 4(np_2)^{\frac{2}{3}} \log^{-\frac{11}{6}} n \leq (np_2 \log n)^{\frac{5}{7}}.$$

Moreover $p_2 \geq \log^{21} n/n$ and H_1 is u-downjumping and so $\sqrt{np_2}$-downjumping by (5). Since R_2 is p_2-pseudorandom we may therefore apply Corollary 16 to find a regular graph H_1' of degree $\Delta(H_1)$ with $H_1 \subseteq H_1' \subseteq H_1 \cup R_2$.

Next, we wish to apply Lemma 17 in order to cover H_1' with edge-disjoint Hamilton cycles. Note that for every $1 \leq j \leq 2m_2 + 1$

$$(9) \qquad np_{(2,j)} \geq \frac{np_2}{(10^{10}+1)m_2 + 1} \geq \frac{(np)^{\frac{3}{4}} \log^{\frac{7}{2}} n \log\log n}{10^{11} \log n} \geq (np)^{\frac{3}{4}} \log^{\frac{5}{2}} n.$$

So we can apply Lemma 17 with G, H_1', $G_{(2,1)}, \ldots, G_{(2,2m_2+1)}$ and $H_{(2,1)}, \ldots, H_{(2,2m_2+1)}$ playing the roles of G, H_0, $G_1, \ldots, G_{2m+1}$ and $H_1, \ldots, H_{2m+1}$ to obtain a collection $\mathcal{HC}_1$ of edge-disjoint Hamilton cycles such that the union $HC_1 := \bigcup \mathcal{HC}_1$ of these Hamilton cycles satisfies

$$H_1' \subseteq HC_1 \subseteq H_1' \cup \bigcup_{j=1}^{2m_2+1} H_{(2,j)} \subseteq H_1' \cup G_2.$$

Write $H_2 := (G_2 \cup R_2) \setminus E(HC_1)$ for the uncovered remainder of $G_2 \cup R_2$. Note that

(HC1) no edge of G incident to x_0 is covered more than once in $\mathcal{HC}_1$;

(HC1′) HC_1 contains no edges from F_1.

Our next aim is to extend H_2 into a regular graph H_2' using some of the edges of R_2'. We will then use some of the edges of G_3 in order to find edge-disjoint Hamilton cycles which cover H_2'. Note that

$$(10) \qquad\qquad d_{H_2}(x) = d_{H_1}(x) + d_{R_2 \cup G_2}(x) - 2|\mathcal{HC}_1|$$

for all $x \in V(G)$. Together with the fact that H_1 is u-downjumping this implies that for all $x \neq x_0$ we have

$$d_{H_2}(x_0) - d_{H_2}(x) = (d_{H_1}(x_0) - d_{H_1}(x)) + (d_{R_2 \cup G_2}(x_0) - d_{R_2 \cup G_2}(x))$$

$$\geq u - (\Delta(R_2) + \Delta(G_2) - (\delta(R_2) + \delta(G_2)))$$

$$\geq u - 8\sqrt{np_2 \log n} \overset{(5)}{\geq} \sqrt{np_2}.$$

(For the second inequality we used the fact that both R_2 and G_2 are p_2-pseudorandom together with (P4) and (P5).) Thus x_0 is the unique vertex of maximum degree in H_2 and H_2 is $\sqrt{np_2}$-downjumping. If $\Delta(H_2)$ is odd, let H_2' be obtained from H_2 by adding some edge from $F_1 \setminus F^*$. Condition (g) ensures that we can choose this edge in such a way that it is not incident to the unique vertex of minimum degree in the $(3p_2 + p_3)$-pseudorandom graph $R_2 \cup G_2 \cup R_2' \cup G_3$. Let

F_1' be the set consisting of this edge. If $\Delta(H_2)$ is even, let $H_2' := H_2$ and $F_1' := \emptyset$. In both cases, let $F_2 := F_1 \setminus F_1'$ and note that H_2' is still $\sqrt{np_2}$-downjumping. Moreover,

$$
\begin{aligned}
\Delta(H_2') - \delta(H_2') \quad &\leq \quad \Delta(H_2) - \delta(H_2) + 1 \\
&\overset{(10)}{\leq} \quad \Delta(H_1) + \Delta(G_2) + \Delta(R_2) - \delta(H_1) - \delta(G_2) - \delta(R_2) + 1 \\
&\leq \quad \Delta(G_1) + \Delta(G_2) + \Delta(R_2) - \delta(G_1) - \delta(G_2) - \delta(R_2) + 1 \\
&\leq \quad 4\sqrt{np_1 \log n} + 8\sqrt{np_2 \log n} + 1 \leq 5\sqrt{np \log n} \\
&\leq \quad (np_2 \log n)^{\frac{5}{7}}.
\end{aligned}
$$

(For the fourth inequality we used the facts that G_1 is p_1-pseudorandom and both R_2 and G_2 are p_2-pseudorandom together with (P4) and (P5). The final inequality follows similarly as (8).) Furthermore, note that $E(H_2') \cap E(R_2') \subseteq F_1'$ and so $H_2' - x_0$ and $R_2' - x_0$ are edge-disjoint. Thus we may apply Corollary 16 to find a regular graph H_2'' of degree $\Delta(H_2')$ with $H_2' \subseteq H_2'' \subseteq H_2' \cup R_2'$. Since x_0 is of maximum degree in H_2', we have the following:

No edge from R_2' incident to x_0 was added to H_2' in order to obtain H_2''.   (†)

Let $G_2^* := (R_2 \cup G_2 \cup R_2' \cup G_3) + F_1'$. Our choice of F_1' and condition (f) together ensure that we can apply Lemma 8 with $R_2 \cup G_2 \cup R_2' \cup G_3$ and F_1' playing the roles of G and F to see that G_2^* is $(3p_2 + p_3)$-pseudorandom. Note that for every $1 \leq j \leq 2m_3 + 1$

$$
np_{(3,j)} \geq (4np_2)^{\frac{3}{4}} \log^{\frac{5}{2}} n \geq (n(3p_2 + p_3))^{\frac{3}{4}} \log^{\frac{5}{2}} n,
$$

where the first inequality follows similarly as (9). Hence we may apply Lemma 17 with G_2^*, H_2'', $G_{(3,1)}, \ldots, G_{(3,2m_3+1)}$ and $H_{(3,1)}, \ldots, H_{(3,2m_3+1)}$ playing the roles of G, H_0, $G_1, \ldots, G_{2m+1}$ and $H_1, \ldots, H_{2m+1}$ to obtain a collection $\mathcal{HC}_2$ of edge-disjoint Hamilton cycles such that the union $HC_2 := \bigcup \mathcal{HC}_2$ of these Hamilton cycles satisfies

$$
H_2'' \subseteq HC_2 \subseteq H_2'' \cup \bigcup_{j=1}^{2m_3+1} H_{(3,j)} \subseteq H_2'' \cup G_3.
$$

We now have the following properties:

(HC2) no edge of G incident to x_0 is covered more than once in $\mathcal{HC}_1 \cup \mathcal{HC}_2$;

(HC2$'$) $HC_1 \cup HC_2$ contains no edges from F_2;

(HC2$''$) $\mathcal{HC}_1 \cup \mathcal{HC}_2$ covers all edges in $(G_1 - F_2) \cup G_2 \cup R_2$.

Indeed, to see (HC2), first note that (†) implies that all edges incident to x_0 in HC_2 are contained in $H_2' \cup G_3$ and thus in $(H_2 + F_1') \cup G_3$, which is edge-disjoint from HC_1. Now (HC2) follows from (HC1) together with the fact that the Hamilton cycles in $\mathcal{HC}_2$ are pairwise edge-disjoint.

Write $H_3 := G_3 \setminus E(HC_2)$ for the subgraph of G_3 which is not covered by the Hamilton cycles in $\mathcal{HC}_2$. Our final aim is to extend H_3 into a regular graph H_3' using some of the edges of R_4. We will then use the edges of G_4 in order to

find edge-disjoint Hamilton cycles which cover H_3' (and thus the edges of G_3 not covered so far). Note that for all $x \in V(G)$

$$d_{H_3}(x) = d(H_2'') + d_{G_3}(x) - 2|\mathcal{HC}_2|.$$

Together with the fact that G_3 is p_3-pseudorandom this implies that

$$(11) \qquad \Delta(H_3) - \delta(H_3) = \Delta(G_3) - \delta(G_3) \overset{\text{(P4),(P5)}}{\leq} 4\sqrt{np_3 \log n}.$$

Thus we can add a set $F_2' \subseteq F_2 \setminus F^*$ of edges at x_0 to H_3 to ensure that x_0 is the unique vertex of maximum degree in the graph H_3' thus obtained from H_3, that H_3' is $\sqrt{np_4}$-downjumping, $\Delta(H_3')$ is even and such that

$$(12) \qquad |F_2'| \leq 4\sqrt{np_3 \log n} + \sqrt{np_4} + 1 \leq 5\sqrt{np_3 \log n} \leq \sqrt{np_2 \log n} \overset{(5)}{\leq} \frac{u}{\log n}.$$

Note that $|F_2 \setminus F^*| = |F_1 \setminus (F_1' \cup F^*)| \geq 2u - 2$ by (7) and since $|F^*| \leq u$ by (6). So we can indeed choose such a set F_2'. Moreover, condition (g) ensures that we can choose F_2' in such a way that it contains no edge which is incident to the unique vertex of minimum degree in the $(p_3' + p_3 + 2p_4)$-pseudorandom graph $G_3' \cup G_3 \cup R_4 \cup G_4$. Let $F_3 := F_2 \setminus F_2'$ and note that

$$\Delta(H_3') - \delta(H_3') \leq \Delta(H_3) - \delta(H_3) + \sqrt{np_4} + 1 \overset{(11)}{\leq} 5\sqrt{np_3 \log n} = 5(np_4)^{\frac{2}{3}} \log^{-\frac{11}{6}} n$$
$$\leq (np_4 \log n)^{\frac{5}{7}}.$$

Furthermore, $E(H_3') \cap E(R_4) \subseteq F_2'$ and so $H_3' - x_0$ and $R_4 - x_0$ are edge-disjoint. Since also $p_4 \geq \log^{21} n/n$, we may apply Corollary 16 to obtain a regular graph H_3'' of degree $\Delta(H_3')$ such that $H_3' \subseteq H_3'' \subseteq H_3' \cup R_4$. Note that since x_0 is of maximum degree in H_3', we have the following:

No edge from R_4 incident to x_0 was added to H_3' in order to obtain H_3''. $\quad (\star)$

Let $G_3^* := (G_3' \cup G_3 \cup R_4 \cup G_4) + F_2'$. Since $|F_2'| \leq 5\sqrt{np_3 \log n} = \sqrt{np_3' \log n}/8$ by (12), we may apply Lemma 8 with $G_3' \cup G_3 \cup R_4 \cup G_4$ and F_2' playing the roles of G and F to see that G_3^* is $(p_3' + p_3 + 2p_4)$-pseudorandom.

Note that for every $1 \leq j \leq 2m_4 + 1$

$$np_{(4,j)} \geq (4np_3')^{\frac{3}{4}} \log^{\frac{5}{2}} n \geq (n(p_3' + p_3 + 2p_4))^{\frac{3}{4}} \log^{\frac{5}{2}} n,$$

where the first inequality follows similarly as (9). Recall that F^* denotes the set of all those edges of G_4 which are incident to x_0. Since $F_2' \cap F^* = \emptyset$, H_3'' and G_4 are edge-disjoint (and so $H_3'', H_{(4,1)}, \ldots, H_{(4,2m_4+1)}$ are pairwise edge-disjoint). Thus we can apply Lemma 17 with $G_3^*, H_3'', G_{(4,1)}, \ldots, G_{(4,2m_4+1)}$ and $H_{(4,1)}, \ldots, H_{(4,2m_4+1)}$ playing the roles of $G, H_0, G_1, \ldots, G_{2m+1}$ and $H_1, \ldots, H_{2m+1}$ to obtain a collection $\mathcal{HC}_3$ of edge-disjoint Hamilton cycles such that the union $HC_3 := \bigcup \mathcal{HC}_3$ of these Hamilton cycles satisfies

$$H_3'' \subseteq HC_3 \subseteq H_3'' \cup \bigcup_{j=1}^{2m_4+1} H_{(4,j)} \subseteq H_3'' \cup G_4.$$

We claim that no edge of G incident to x_0 is covered more than once in $\mathcal{HC} :=$ $\mathcal{HC}_1 \cup \mathcal{HC}_2 \cup \mathcal{HC}_3$. Indeed, (HC2) implies that this was the case for $\mathcal{HC}_1 \cup \mathcal{HC}_2$. Moreover, recall that the Hamilton cycles in $\mathcal{HC}_3$ are pairwise edge-disjoint. In addition, ($\star$) implies that all edges incident to x_0 in HC_3 are contained in

$$H_3' + F^* = H_3 + F_2' + F^* \subseteq H_3 + F_2.$$

So (HC2$'$) implies that none of these edges lies in $HC_1 \cup HC_2$, which proves the claim.

Note that (HC2$''$) and the definition of $\mathcal{HC}_3$ together imply that $\mathcal{HC}$ covers all edges of $G - F_3$. Let $F \subseteq F_3$ be the set of uncovered edges. Then F and $\mathcal{HC}$ are as required in the lemma. $\qquad\square$

We remark that for the final application of Lemma 17 in the proof of Lemma 19 it would have been enough to consider $G_3 \cup R_4 \cup G_4$ instead of $G_3' \cup G_3 \cup R_4 \cup G_4$ (since H_3'' and all the $G_{(4,j)}$ are contained in $(G_3 \cup R_4 \cup G_4) + F_2'$). However, we would not have been able to apply Lemma 8 in this case since $|F_2'| > \sqrt{np_3 \log n}/8$. Introducing G_3' ensures that the conditions of Lemma 8 are satisfied (and this is the only purpose of G_3').

We can now combine Theorem 12 and Lemma 19 in order to prove Theorem 1.

Proof of Theorem 1. Lemma 19 implies that a.a.s. G contains a collection $\mathcal{HC}$ of Hamilton cycles and a collection F of edges incident to the unique vertex x_0 of maximum degree such that no edge of G incident to x_0 is contained in more than one Hamilton cycle in $\mathcal{HC}$ and such that the Hamilton cycle in $\mathcal{HC}$ cover precisely the edges of $G - F$. Moreover, by Theorem 12, a.a.s. $G - x_0$ is Hamilton-connected.

If $|F|$ is odd, we add one edge of $G - F$ incident to x_0 to F. We still denote the resulting set of edges by F. Let $r := |F|/2$ and $e_1 e_1', \ldots, e_r e_r'$ be pairs of edges such that F is the union of all these $2r$ edges. Since $G - x_0$ is Hamilton-connected there exists a Hamilton cycle C_i of G containing both e_i and e_i'. Then $\mathcal{HC} \cup \{C_1, \ldots, C_r\}$ is a collection of $\lceil \Delta(G)/2 \rceil$ Hamilton cycles covering G, as desired. $\qquad\square$

Using further iterations in the proof of Lemma 19, one could reduce the exponent 117 in Lemma 19 (and thus in Theorem 1). One further iteration would lead to an exponent of 60, while the effect of yet further iterations quickly becomes insignificant.

References

[1] S. Ben-Shimon, M. Krivelevich and B. Sudakov, On the resilience of Hamiltonicity and optimal packing of Hamilton cycles in random graphs, *SIAM J. Discrete Mathematics* **25** (2011), 1176–1193.

[2] B. Bollobás, The evolution of sparse graphs, *Graph Theory and Combinatorics*, Academic Press, London (1984), 35–57.

[3] B. Bollobás, *Random Graphs*, Academic Press, London, 1985.

[4] B. Bollobás and A. Frieze, On matchings and Hamiltonian cycles in random graphs. *Random graphs '83* (Poznan, 1983), North-Holland Math. Stud., 118, North-Holland, Amsterdam (1985), 23–46.

[5] A. Frieze and M. Krivelevich, On packing Hamilton cycles in ε-regular graphs, *J. Combin. Theory B* **94** (2005), 159–172.

[6] A. Frieze and M. Krivelevich, On two Hamilton cycle problems in random graphs, *Israel J. Math.* **166** (2008), 221–234.

[7] R. Glebov, M. Krivelevich and T. Szabó, On covering expander graphs by Hamilton cycles, preprint.

[8] D. Hefetz, M. Krivelevich and T. Szabó, Hamilton cycles in highly connected and expanding graphs, *Combinatorica* **29** (2009), 547–568.

[9] S. Janson, T. Łuczak and A. Ruciński, *Random graphs*, Wiley-Interscience, 2000.

[10] F. Knox, D. Kühn and D. Osthus, Approximate Hamilton decompositions of random graphs, *Random Structures & Algorithms* **40** (2012), 133–149.

[11] F. Knox, D. Kühn and D. Osthus, Edge-disjoint Hamilton cycles in random graphs, preprint.

[12] M. Krivelevich and W. Samotij, Optimal packings of Hamilton cycles in sparse random graphs, preprint.

[13] D. Kühn and D. Osthus, Hamilton decompositions of regular expanders: a proof of Kelly's conjecture for large tournaments, preprint.

[14] D. Kühn and D. Osthus, Hamilton decompositions of regular expanders: applications, preprint.

[15] V. Rödl, On a packing and covering problem, *European J. Combin.* **6** (1985), 69–78.

[16] A. Thomason, Pseudo-random graphs, *Annals of Discrete Mathematics* **33** (1987) 307–331.

[17] W. Tutte, The factors of graphs, *Canad. J. Math.* **4** (1952), 314–328.

[18] W. Tutte, A short proof of the factor theorem for finite graphs, *Canad. J. Math.* **6** (1954), 347–352.

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